JOURNAL OF APPROXIMATION THEORY **40**, 158–160 (1984)

Linear Independence of Translates of a Box Spline

RONG-QING JIA

Mathematics Department,
University of Wisconsin, Madison, Wisconsin 53706, U.S.A.

Communicated by Charles A. Micchelli

Received October 28, 1982

A box spline M_Ξ is a distribution on $\mathbb{R}^n$ given by the rule

$$M_\Xi = \phi \mapsto \int_{[0,1]^n} \phi\left(\sum_{i=1}^{n} \lambda(i)\,\xi_i\right) d\lambda$$

for some sequence $\Xi := (\xi_i)_1^n$ (see [1]). We think of Ξ as a set of cardinality $|\Xi| = n$. For a subset V of $\mathbb{R}^m$, we are interested in linear independence of the translates $M_{\Xi,v} := M_\Xi(\cdot - v)$, $v \in V$. By linear independence of $(M_{\Xi,v})_V$ we mean that the linear map

$$a \mapsto \sum_v a(v)\, M_{\Xi,v}$$

is one-to-one on $\mathbb{R}^V$ (see [1]). For the special case $\Xi \subseteq V = \mathbb{Z}^m$, de Boor and Höllig show that $(M_{\Xi,v})_{v\in V}$ is linearly dependent unless

$$|\det Z| = 1 \qquad \text{for all bases } Z \subseteq \Xi$$

(see Proposition 4 of [1]). In this note we shall prove that the converse of the above statement is also true. Thus we have the following

THEOREM. *Suppose that $\Xi \subseteq V = \mathbb{Z}^m$, and that $\langle \Xi \rangle$, the affine hull of Ξ, is the whole of $\mathbb{R}^m$. Then $(M_{\Xi,v})_V$ is linearly independent if and only if*

$$|\det Z| = 1 \qquad \textit{for all bases } Z \subseteq \Xi.$$

Proof. We only need to prove the "if" part.

The proof proceeds by induction on $|\Xi|$. The case $|\Xi| = 1$ is trivial. Suppose now that the theorem has been proved for any Ξ' with $|\Xi'| < |\Xi|$.

Without loss of generality, we may assume Ξ contains all the unit vectors, i.e.,

$$\{e_1, \ldots, e_m\} \subset \Xi.$$

0021-9045/84 $3.00

Indeed, Ξ contains some basis $Z = \{\zeta_1, ..., \zeta_m\}$ for $\mathbb{R}^m$, and, by assumption, $|\det Z| = 1$. Let Q be the linear map which carries ζ_i to $e_i (i = 1, ..., m)$. Since $|\det Z| = 1$, Q must map V to V. Moreover, $|\det Q| = 1$. Hence $Q\Xi \subseteq V$ and (cf. [1])

$$M_\Xi = |\det Q| \, M_{Q\Xi} \circ Q = M_{Q\Xi} \circ Q.$$

It follows that $(M_{\Xi, v})_V$ is linearly independent if $(M_{Q\Xi, v})_V$ is. Thus, if necessary, we can work with $M_{Q\Xi}$ instead of M_Ξ.

In the following we divide our consideration into two cases.

Case 1. There exists some e_k such that $\langle e_k \rangle \cap \langle \Xi \backslash e_k \rangle = 0$.

Without loss we may assume $\langle e_m \rangle \cap \langle \Xi \backslash e_m \rangle = 0$. Then $\langle \Xi \backslash e_m \rangle = \mathbb{R}^{m-1}$. Any $v \in V$ can be uniquely written as

$$v = je_m + v' \qquad \text{with} \quad j \in Z \text{ and } v' \in \mathbb{Z}^{m-1}.$$

It is easily seen that

$$M_\Xi(x) = M_{e_m}(x_m) \, M_{\Xi \backslash e_m}(x')$$

where $x = (x_1, ..., x_{m-1}, x_m)$ and $x' = (x_1, ..., x_{m-1})$. Suppose now that $\sum_{v \in V} a(v) \, M_\Xi(\cdot - v) = 0$ for some $a \in \mathbb{R}^V$. Then

$$\sum_{j \in Z} \sum_{v' \in \mathbb{Z}^{m-1}} a(je_m + v') \, M_{e_m}(x_m - je_m) \, M_{\Xi \backslash e_m}(x' - v') = 0$$
$$\text{for any} \quad x_m \in \mathbb{R} \text{ and } x' \in \mathbb{R}^{m-1}. \tag{1}$$

Set $x_m = (i + \frac{1}{2}) \, e_m$ in (1). We obtain

$$\sum_{v' \in \mathbb{Z}^{m-1}} a(ie_m + v') \, M_{\Xi \backslash e_m}(x' - v') = 0 \qquad \text{for any} \quad x' \in \mathbb{R}^{m-1} \text{ and } i \in \mathbb{Z}.$$

By induction hypothesis, $(M_{\Xi \backslash e_m, v'})_{v' \in \mathbb{Z}^{m-1}}$ is linearly independent. Therefore

$$a(ie_m + v') = 0 \qquad \text{for all} \quad v' \in \mathbb{Z}^{m-1} \text{ and } i \in \mathbb{Z}.$$

That is, $a = 0$. This proves the linear independence of $(M_{\Xi, v})_V$ in Case 1.

Case 2. The complement of case 1; i.e., $\langle \Xi \backslash e_k \rangle = \mathbb{R}^m$ for every $k = 1, ..., m$.

Suppose that $\sum_{v \in V} a(v) \, M_{\Xi, v} = 0$ for some $a \in \mathbb{R}^V$. Then

$$\sum (\nabla_{e_k} a)(v) \, M_{\Xi \backslash e_k}(\cdot - v)$$
$$= D_{e_k} \left(\sum a(v) \, M_\Xi(\cdot - v) \right) = 0 \qquad \text{for} \quad k = 1, ..., m$$

(see [1]). By induction hypothesis, $(M_{\Xi\backslash e_m,v})_V$ is linearly independent. Hence

$$\nabla_{e_k} a = 0 \qquad \text{for each} \quad k = 1,\ldots, m.$$

It follows that

$$a(v) = a(v - e_k) \qquad \text{for} \quad k = 1,\ldots, m.$$

Therefore

$$a(v) = a(0) \qquad \text{for all} \quad v \in V.$$

Finally,

$$a(0) = a(0) \sum_{v \in V} M_{\Xi,v} = \sum_{v \in V} a(v)\, M_{\Xi,v} = 0.$$

This shows that $(M_{\Xi,v})_V$ is linearly independent. Our proof is complete.

Postscript. After this work was done, I was made aware that W. Dahmen and C. A. Micchelli also prove this theorem, in the paper "Translates of Multivariate Splines," which will appear in *Linear Algebra and Its Applications*. However, the proof given here is particularly simple and elementary.

Reference

1. C. de Boor and K. Höllig, "*B*-splines from Parallelepipeds," Mathematics Research Center Technical Summary Report No. 2320, University of Wisconsin–Madison.